# Lecture 10: Using Symmetries

## Circular Symmetry

- One may compute ~~sepa~~ solutions for the Helmholtz equation in rectangles of higher dimension (see HW), and we will treat the next simplest case: disks in $\mathbb{R}^2$.

- As in HW 1, polar coordinates are $(x_1, x_2) = (r\cos(\theta), r\sin(\theta))$ and
$$\Delta = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$
  - no mixed partials = easy separation

- To solve the radial eigenvalue equation, we will need special functions called Bessel functions, that solve
$$z^2 f''(z) + z f'(z) + (z^2 - k^2) f(z) = 0$$
for $k \in \mathbb{C}$.

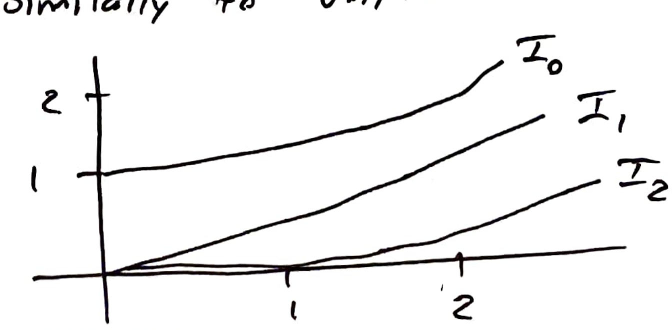
The solutions include a linearly independent pair $J_k(z)$, $Y_k(z)$

For $k \in \mathbb{N}_0$, $J_k(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin(\theta) - k\theta)\,d\theta$
$$= \left(\frac{z}{2}\right)^k \sum_{\ell=0}^\infty \frac{1}{\ell!\,(k+\ell)!}\left(-\frac{z^2}{4}\right)^\ell$$

Notice, as $z \to 0$, $J_k(z)$ has lowest-order term $\left(\frac{z}{2}\right)^k \cdot \frac{1}{k!}$ and so it shrinks similarly to $z^k$. We set $J_{-k} = (-1)^k J_k$.

Similarly, $Y_k(z)$ shrinks like $C_k z^{-|k|}$ for $k \in \mathbb{Z}$.

- The equation is sometimes modified to
$$z^2 f'' + z f' + (z^2 + k^2) f = 0$$
with modified Bessel functions $I_k$, $K_k$ that shrink similarly to $J_k$, $Y_k$.



## Lemma 5.4

Suppose $\phi \in C^2(\mathbb{R}^2)$ solves $-\Delta\phi = \lambda\phi$.

and factors $\phi(r,\theta) = h(r)\omega(\theta)$. Then, up to constant multiplication, $\phi$ has the form

$$\phi_{\lambda,k}(r,\theta) = h_k(r)e^{ik\theta}$$

for some $k \in \mathbb{Z}$ for

$$h_k(r) = \begin{cases} r^{|k|} & \lambda = 0 \\ J_k(r\sqrt{\lambda}) & \lambda > 0 \\ I_k(r\sqrt{-\lambda}) & \lambda < 0 \end{cases}$$

$\boxed{\text{Pf}}$ For $\phi = h\omega$, we obtain

$$\frac{\omega}{r}\frac{\partial}{\partial r}\left(r\frac{\partial h}{\partial r}\right) + h/r^2 \frac{\partial^2\omega}{\partial\theta^2} + \lambda h\omega = 0$$

or

$$\frac{1}{h}\left(r\,\partial/\partial r\right)^2 h + a^2r^2 = -\frac{1}{\omega}\frac{\partial^2\omega}{\partial\theta^2} \quad \text{for } \omega, h \text{ nonzero.}$$

as in lemma 5.1, both sides must equal some constant $\mu$.

The $\theta$-equation is $\quad -\frac{\partial^2\omega}{\partial\theta^2} = \mu\omega$

and $\omega(\theta)$ is $2\pi$-periodic (polar coordinates).

As in thm 5.2 (previous lecture), this can only happen for $\mu = k^2$ for $k \in \mathbb{Z}$, giving solutions

$$\omega_k(\theta) = e^{ik\theta}$$

we focus on $\phi$ first.

- To help with the radial equation, we must have/impose a boundary condition at $r=0$. Since $\phi \in C^2$, since $r = \sqrt{x_1^2 + x_2^2}$ is not differentiable at $(0,0)$.

Notice that $re^{\pm i\theta} = x_1 \pm ix_2$ are $C^\infty$, so for $k \in \mathbb{Z}$, we look at

$$r^{|k|}e^{ik\theta} = \begin{cases} (x_1+ix_2)^k & k \in \mathbb{N}_0 \\ (x_1 - ix_2)^{-k} & -k \in \mathbb{N} \end{cases}$$

which are also $C^\infty$.

Differentiability of $\phi$ at the origin requires $h_k(r)$ to act like $ar^{|k|}$ for some $a$ as $r \to 0$.

- The radial component of the PDE is

$$(A) \quad \left(r\frac{\partial}{\partial r}\right)^2 h_K + (\lambda r^2 - K^2) h_K = 0$$

if $\lambda = 0$, we have $r\frac{\partial h_K}{\partial r} + r^2 \frac{\partial^2 h_K}{\partial r^2} - K^2 h_K = 0$, which is homogeneous in $r$ & solved by $h_K = r^\alpha$ for $\alpha \in \mathbb{R}$. ~~As~~

~~giving~~

Trying this guess gives $(\alpha^2 - K^2) h_K = 0$, so $\alpha = \pm K$.

As a second-order ODE, we have independent solutions $r^{\pm K}$,

For $K=0$, we have $1$ & $\ln(r)$. By our "Boundary Condition",

we rule out $\ln(r)$ & $r^{-|K|}$ to give

$$h_K(r) = r^{|K|}$$

with resulting solutions $\Phi_{0,K}(r,\theta) = r^{|K|} e^{iK\theta}$.

- For $\lambda > 0$, (A) may be changed into the Bessel equation using the change of variables $z = r\sqrt{\lambda}$, $\frac{\partial}{\partial z} = \sqrt{\lambda}\frac{\partial}{\partial r}$

So

$$r^2 \frac{\partial^2}{\partial r^2} h_K + r\frac{\partial}{\partial r} h_K + (\lambda r^2 - K^2) h_K = 0$$

$$\Updownarrow$$

$$z^2 \frac{\partial^2}{\partial z^2} h_K + z\frac{\partial}{\partial z} h_K + (z^2 - K^2) h_K = 0$$

Since the boundary condition rules out $Y_K(r\sqrt{\lambda})$, but $J_K(r\sqrt{\lambda})$ satisfies it, $h_K(r) = J_K(r\sqrt{\lambda})$ giving

$$\Phi_{\lambda,K}(r,\theta) = J_K(r\sqrt{\lambda}) e^{iK\theta}$$

and as $J_K(r\sqrt{\lambda}) \sim (r\sqrt{\lambda})^K$ as $r \to 0$, $\Phi_{\lambda,K}$ is appropriately defined, and actually $C^2$. The power series expansion actually gives $\Phi_{\lambda,K}$ is $C^\infty$.

- For $\lambda > 0$, $z = r\sqrt{-\lambda}$ gives a similar breakdown with $I_K$.

$\square$

ex.) The vibration of a drumhead may be modeled by the wave equation on domain $\mathbb{D} = B(0;1) \subseteq \mathbb{R}^2$.

This reduces, by Lemma 5.1, to solving the Helmholtz equation as above. We have a boundary condition $\Phi(1,\theta) = 0$ or $h_K(1) = 0$. This rules out $\lambda \leq 0$ (because $h$ has no zeroes for $r > 0$ in this case). Then, $h_K = J_K(r\sqrt{\lambda})$ with B.C. $J_K(\sqrt{\lambda}) = 0$.

There are infinitely many zeroes of $J_K$, which we write as

$$0 < j_{K,1} < j_{K,2} \cdots$$

Restricting $\lambda$ to these values gives $\lambda_{K,m} = j_{K,m}^2$

giving eigenfunctions

$$\Phi_{K,m}(r,\theta) = J_K(j_{K,m} r)\, e^{iK\theta}$$

~ We won't prove this, but this is a complete list of eigenfunctions

• The eigenvalues correspond to vibrational frequencies

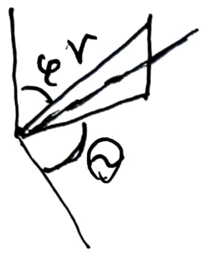
$$w_{K,m} = c\, j_{K,m}$$

- Unlike 1D, the ratios $w_{K,m}/w_{0,1}$ have no clear pattern

~ overtones mix more - no clear frequencies

# Spherical Symmetry

- We use spherical coordinates $(x_1, x_2, x_3) = (r\sin(\varphi)\cos(\theta), r\sin(\varphi)\sin(\theta), r\cos(\varphi))$




- The spherical Laplacian is

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin(\varphi)}\frac{\partial}{\partial\varphi}\left(\sin(\varphi)\frac{\partial}{\partial\varphi}\right) + \frac{1}{r^2\sin^2(\varphi)}\frac{\partial^2}{\partial\theta^2}$$

~ Because coefficients rely on both $\varphi$ & $r$, separation isn't immediately clear

~ However, we may write

$$\Delta = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\left(\frac{1}{\sin(\varphi)}\frac{\partial}{\partial\varphi}\left(\sin(\varphi)\frac{\partial}{\partial\varphi}\right) + \frac{1}{\sin^2(\varphi)}\frac{\partial^2}{\partial\theta^2}\right)$$

$$= 1/r^2\,\partial/\partial r\,(r^2\,\partial/\partial r) + 1/r^2\,\underbrace{\Delta_{S^2}}_{\text{Spherical Laplacian}}$$

~ the spherical Laplacian is the only second-order differential operator invariant under rotations of the sphere, so it arises naturally in other contexts.

- Let us first focus on the Helmholtz Problem on the sphere to separate $\varphi$ & $\theta$. The arising ODE is the associated Legendre equation

$$(1-z^2)f''(z) - 2zf'(z) + \left(\nu(\nu+1) - \frac{\mu^2}{1-z^2}\right)f(z) = 0$$

with parameters $\mu, \nu \in \mathbb{C}$. A pair of linearly indep. solutions is given by the Legendre functions

$P_\nu^\mu(z)$ & $Q_\nu^\mu(z)$

In the special case $\nu = \ell \in \mathbb{N}_0$ & $\mu \in \{-\ell, -\ell+1, \dots 0 \dots \ell-1, \ell\}$

$$P_\ell^m(z) = \frac{(-1)^m}{2^\ell \ell!}(1-z)^{m/2} \frac{d^{\ell+m}}{dz^{\ell+m}}(z^2-1)^\ell$$

•) These functions help define Spherical harmonics (used in geometry)

$$Y_\ell^m(\varphi, \theta) = C_{m,\ell}\, e^{im\theta} P_\ell^m(\cos(\varphi))$$

for a constant $C_{m,\ell}$.

For $z = \cos(\varphi)$, $1-z^2 = \sin^2(\varphi)$, $Y_\ell^m$ is a degree-$\ell$ polynomial in $\sin(\varphi)$, $\cos(\varphi)$ so they $Y_\ell^m(\varphi, \theta)$ is smooth in $S^2$.

$\boxed{\text{Lemma 5.6}}$ Suppose $u \in C^2(S^2)$ solves $-\Delta_{S^2} u = \lambda u$. and factors as $u(\varphi, \theta) = v(\varphi) w(\theta)$. Then, up to a multiplicative constant, $u = Y_\ell^m$ for $\ell \in \mathbb{N}_0$, $m \in \{-\ell \dots \ell\}$, with corresponding eigenvalue $\lambda_\ell = \ell(\ell+1)$ (with multiplicity $2\ell+1$).

$\boxed{\text{Pf}}$ $u = vw$ leads to the separated equation

$$\frac{\sin(\varphi)}{v} \frac{\partial}{\partial \varphi}\left(\sin(\varphi) \frac{\partial v}{\partial \varphi}\right) + \lambda = -\frac{1}{w} \frac{\partial^2 w}{\partial \theta^2}$$

The continuity of $u$ requires $w$ to be $2\pi$-periodic so

$$-\frac{\partial^2 w}{\partial \theta^2} = \mu w$$ has a full set of solutions $w(\theta) = e^{im\theta}$

for $\mu = m^2$, $m \in \mathbb{Z}$.

For $u(\theta, \varphi) = v_m(\varphi) e^{im\theta}$, $-\Delta_{S^2} u = \lambda u$ becomes

$$\frac{1}{\sin(\varphi)} \frac{d}{d\varphi}\left(\sin(\varphi) \frac{dv_m}{d\varphi}\right) + \left(\lambda - \frac{m^2}{\sin^2(\varphi)}\right) v_m = 0$$

we substitute $z = \cos(\varphi)$, $v_m(\varphi) = f(\cos(\varphi))$ and obtain

$$(1-z^2) f'' - 2z f' + \left(\lambda - \frac{m^2}{1-z^2}\right) f = 0,$$

the Legendre equation with $m = \mu$, $\lambda = \nu(\nu+1)$.

- Due to the use of Spherical coordinates, we create artificial boundaries at $\varphi = \pi$, $\varphi = 0$ (poles of the sphere) and our solution must be Smooth at these.

$Q_\nu^m(z)$ diverges as $z \to 1$ or as $\varphi \to 0$ for any $\nu$

Except for the Special cases $P_\ell^m(z)$, $P_\nu^m(z)$ diverges as $z \to -1$ (or $\varphi \to +\pi$). Thus, $v_m(\varphi) = P_\ell^m(\cos(\varphi))$ for some $\ell \in \mathbb{N}_0$, $|m| \le \ell$, So that $u$ is proportional to $Y_\ell^m$.

Since $\lambda = \nu(\nu+1) = \ell(\ell+1)$ and $m \in \{-\ell, \ldots, \ell\}$, we have the eigenvalue claims. $\square$

RmK: This is, again, a complete set of eigenfunctions for $\Delta_{S^2}$.

ex.) Schrödinger's Quantum model for a hydrogen atom says that electron energy levels are given by the eigenvalues of

$$(-\Delta - 1/r)\phi = \lambda \phi$$

on $\mathbb{R}^3$.

- we assume the eigenfunctions are bounded near $r=0$ and decaying to $0$ as $r \to \infty$.
- we next separate:

$$-\tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(r^2 \tfrac{\partial}{\partial r}\right)\phi - \tfrac{1}{r^2}\Delta_{S^2}\phi - \tfrac{1}{r}\phi = \lambda\phi$$

$$\Leftrightarrow \quad \Delta_{S^2}\phi = -\tfrac{\partial}{\partial r}\left(r^2 \tfrac{\partial}{\partial r}\phi\right) - r\phi - \lambda r^2 \phi$$

& set $\phi = h(r)w(\varphi,\theta)$.

$$\tfrac{1}{w}\Delta_{S^2} w = \tfrac{1}{h}\left(-\tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial}{\partial r}h\right) - rh - \lambda r^2 h\right)$$

By Lemma 5.6 above, the angular components are $Y_\ell^m$ and $\phi = h(r)Y_\ell^m(\varphi,\theta)$ (note $K = \ell(\ell+1)$).

The radial equation is then

$$\left[-\tfrac{1}{r^2}\tfrac{\partial}{\partial r}\left(r^2\tfrac{\partial}{\partial r}\right) + \frac{\ell(\ell+1)}{r^2} - \frac{1}{r}\right]h(r) = \lambda h(r)$$

and we must analyze this ODE.

$$\left[ -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\right) + \frac{\ell(\ell+1)}{r^2} - \frac{1}{r} \right] h(r) = \lambda h(r) \qquad (\star)$$

- One analysis strategy is to first consider asymptotic behavior as $r \to \infty$ or $r \to 0$.

~ Heuristic/to produce a guess ~

- Suppose we assume $h(r) \sim r^\alpha$ as $r \to 0$ (shrinks like $r^\alpha$).

Plugging into $(\star)$ and comparing sides

$$\lambda r^\alpha = -\alpha(\alpha+1) r^{\alpha-2} + \ell(\ell+1) r^{\alpha-2} - r^{\alpha-1}$$

this gives $h(r) \sim r^\alpha$ as $r \to 0$ is possible only if

$$-\alpha(\alpha+1) = \ell(\ell+1) \qquad \text{(to eliminate } r^{\alpha-2} \text{ on the RHS)}$$

Since $h(r)$ can't diverge, we can only have $\alpha = \ell$

So $h(r) \sim r^\ell$ as $r \to 0$

As $r \to \infty$, we consider terms in $(\star)$ of order $r^0$ and drop the rest, giving

$$-h''(r) \sim \lambda h(r)$$

If $\lambda \geq 0$, then $h(r)$ couldn't decay at infinity, so we set ~~...~~ $\lambda < 0$ and instead note

$$h(r) \sim c e^{-\sqrt{-\lambda}\, r} \quad \text{as } r \to \infty.$$

- This gives us a guess of a form $h(r) = q(r) r^\ell e^{-\sqrt{-\lambda}\, r}$ with the conditions $q(0) = 1$, & $q$ grows more slowly than an exponential as $r \to \infty$.

- Substituting this guess into $(\star)$ gives for $\delta^2 = -\lambda$

$$r q'' + 2(1+\ell - r\delta) q' + (1 - 2\delta(\ell+1)) q = 0 \qquad (\star\star)$$

- This is still difficult to solve.

• Suppose $q(r)$ is given by $q(r) = \sum_{k=0}^{\infty} a_k r^k$

with $a_0 = 1$.

Plugging this into (AA) gives

$$0 = \sum_{k=0}^{\infty} \left[ k(k-1)a_k r^{k-1} + 2(\ell + 1 - r\sigma) k a_k r^{k-1} + (1 - 2\sigma(\ell+1)) a_k r^k \right]$$

~ We equate the coefficient of each $r^k$ to 0 and notice

$$a_{k+1} = \frac{2\sigma(k+\ell+1) - 1}{(k+1)(k+2\ell+2)} a_k$$

this recursion means $a_k \sim (2\sigma)^k / k!$, giving $q(r) \sim c e^{2\sigma r}$ and violating our necessary growth.

• The only way to avoid this is to have the $a_k$ terminate: $q(r) = \sum_{k=0}^{m} a_k r^k$ is then a polynomial.

for this to occur, $2\sigma(k+\ell+1) - 1 = 0$ or (so $a_{k+1} = 0$)
$\sigma = \frac{1}{2(k+\ell+1)}$ for some $k$

This restricts eigenvalues to $\lambda_n = -1/4n^2$ for $n \in \mathbb{N}$.
While our argument is a bit "loose", this is actually a complete set of eigenvalues corresponding to eigenfunctions

$$\phi_{n,\ell,m}(r, \varphi, \theta) = r^\ell q_{n,\ell}(r) e^{-r/2n} Y_\ell^m(\varphi, \theta)$$

for $\ell \in \{0, \dots, n-1\}$, $m \in \{-\ell, \dots, \ell\}$ with coefficients computed & $q_{n,\ell}(r)$ the polynomial with coefficients computed by the recursion above.

~ this gives a theoretical explanation of the emission spectrum of hydrogen gas!